On logarithmic norms for differential algebraic equations

Renate Winkler
Humboldt-University Berlin
Unter den Linden 6, D-10099 Berlin, Germany

Abstract

Logarithmic matrix norms are well known in the theory of ordinary differential equations (ODEs) where they supply estimates for error growth and the growth of the solutions. In this paper we present a natural generalization of logarithmic norms which makes it applicable to differential-algebraic equations (DAEs) and yield sense-full estimates for the growth of the solutions of the DAE. As there are various possibilities we show how they correspond to each other.

1 Introduction

We want to present und discuss possibilities for generalizing the concept of logarithmic norms to make it applicable to DAEs. Logarithmic norms for matrices were introduced in 1958 by Dahlquist and Lozinskij [1],[5]. For a square matrix $A \in L(\mathbb{R}^m)$ the logarithmic matrix norm, or for short the logarithmic norm, is defined by

$$\mu[A] = \lim_{h \to 0+} \frac{||I + hA|| - 1}{h}, \tag{1.1}$$

where $||\cdot||$ denotes a matrix norm induced by a vector norm $|\cdot|$. In general, the logarithmic matrix norm depends on the used vector norm $|\cdot|$. If $|\cdot|_2$ denotes the Euklidian vector norm, E is a nonsingular matrix, and $|\cdot|_E$ denotes the vector norm with $|x|_E = |Ex|_2$ then we have for the induced matrix norm $||A||_E = ||EAE^{-1}||_2$ and ,analogously, for the corresponding logarithmic matrix norm

$$\mu_E[A] = \mu_2[EAE^{-1}].$$

$\mu[\cdot]$ can take values in $\mathbb{R}$ and is not a usual matrix norm. It can be written as

$$\mu[A] = \lim_{h \to 0+} \frac{ln||e^{hA}||}{h}, \tag{1.2}$$

and, if $|\cdot|$ is an inner product norm $|\cdot|^2 = <\cdot,\cdot>$, also as

$$\mu[A] \;=\; \max_{z\neq\ 0} \frac{< Az, z >}{|z|^2}. \tag{1.3}$$

To illustrate the use of logarithmic matrix norms in the theory of ODEs we follow the lines of [2] and quote the following theorem of Dahlquist [1] :

Theorem 1.1 *Let x and $\bar{x}$ be solutions of*

$$x'(t) = f(x(t), t) \quad , \quad t \in [0, T] \tag{1.4}$$

with values $x(t)$, $\bar{x}(t)$ lying in some neighborhood $M_t := \{z \in I\!R^m : |z - h(t)| \leq \psi(t)\}$, where h is some continuous auxiliary function. Let ν be a piecewise continuous scalar function satisfying

$$\mu[f'_x(z,t)] \leq \nu(t),\ \forall t \in [0,T], \forall z \in M_t.$$

Then it holds

$$|x(t_2) - \bar{x}(t_2)| \;\leq\; exp(\int_{t_1}^{t_2} \nu(s)ds)\; |x(t_1) - \bar{x}(t_1)|,$$

for all t_1, t_2 satisfying $0 \leq t_1 \leq t_2 \leq T$.

If $\mu[f'_x(z,t)] \leq 0\, \forall t \in [0,T], \forall z \in M_t$ the differential equation (1.4) is called dissipative. The freedom of choosing an arbitrary vector norm can be exploited to select a vector norm for which $\nu(t)$ is as small as possible. For a detailed representation we refer to [2].

As a conclusion of the theorem 1.1 we obtain for time-dependent linear ODEs the following

Corollary 1.2 *Consider*

$$x'(t) = A(t)x(t) + q(t) \tag{1.5}$$

with continuous coefficients $A(\cdot)$ and continuous $q(\cdot)$. Let denote

$$L(t) = \int_0^t \mu[A(\tau)]d\tau.$$

Then, for a solution $x(\cdot)$ of (1.5) it holds

$$|x(t)| \leq e^{L(t)} \cdot |x(0)| + e^{L(t)} \cdot \int_0^t e^{-L(s)} \cdot |q(s)|ds \qquad \forall t \geq 0.$$

Aiming at similar estimates for the growth of the solutions of DAEs we describe a generalization of the concept of logarithmic norms which was presented by März [6]. Here we take care of the fact that the solutions of DAEs proceed in lower dimensional subspaces. Our way is to use the correspondence of the solutions of DAEs to the solutions of corresponding inherent regular ODEs in invariant sub-manifolds . Linearization then lead to time-dependent linear ODEs with time-dependent invariant subspaces.

This leads us to a concept of logarithmic matrix norms with respect to subspaces which we will describe in chapter 2.It gives useful estimates for the solution of ODEs with invariant solution manifolds. We already mentioned that the logarithmic matrix norm generally depends on the chosen vector norm. But we found the freedom to use different vector norms (corresponding to the use of constant transformations of the solutions) to restrictive to obtain optimal results and to compare results obtained by different approaches . We therefore investigate how the estimates are influenced by time-dependent transformations.

Based on this we are able to derive estimates for the solutions of DAEs. They are presented in chapter 3. We start with linear index-1-DAEs, explain the concept of inherent regular ODEs and show that different choices for the inherent regular ODE lead to systems that represent time-dependent transformations of each other. We show how these results are connected to those obtained by the concept of logarithmic norms for matrix pencils by Higueras/Garcia-Celayeta [4] and deal finally with nonlinear index-1 DAEs, where the state-space-form (see [3]) yields a nonlinear inherent regular ODE.

2 Logarithmic matrix norms with respect to subspaces

Let us consider the linear homogeneous time-dependent ODE

$$
\begin{aligned}
x'(t) &= W(t)x(t) + q(t), \quad t \in \mathbb{R}, && (2.1)\\
x(t_0) &\in U(t_0), && (2.2)
\end{aligned}
$$

where W is a continuous matrix function and $U(t)$ is a subspace of $\mathbb{R}^m$ depending continuously on t in which all solutions of (2.1,2.2) proceed. i.e. $U(t)$ is an invariant subspace of the ODE (2.1).

We are now interested in estimates for the growth of solutions in the invariant subspace $U(t)$. Due to this invariant subspace the estimates obtained here may differ substantially from estimates concerning all solutions of the ODE (1.1).

We start with a definition of an induced matrix norm with respect to some subspace.

Definition 2.1 *For a given vector norm $|\cdot|$ and a subspace $U \subset \mathbb{R}^m$ we define the induced matrix norm with respect to the subspace U by*

$$
\|A\|^U \quad := \quad \max_{z \in U, z \neq 0} \frac{|Az|}{|z|} \qquad \forall A \in L(\mathbb{R}^m).
$$

Based on this we define a logarithmic matrix norm with respect to a subspace.

Definition 2.2 *For a given subspace $U \subset \mathbb{R}^m$ we define the logarithmic matrix norm with respect to the subspace U by*

$$
\mu[A]^U \quad := \quad \lim_{h \to 0^+} \frac{\|I + hA\|^U - 1}{h} \qquad \forall A \in L(\mathbb{R}^m).
$$

Analogously to the usual theory of logarithmic matrix norm one easily justifies the following properties :

- Let $|\cdot|_2$ denote the Euklidian vector norm, E be a nonsingular matrix, and $|\cdot|_E$ denote the vector norm with $|x|_E = |Ex|_2$. Then, it holds

  $$
  \|A\|_E^U = \|EAE^{-1}\|_2^{EU} \quad .
  $$

- Analogously, for the corresponding logarithmic matrix norm it holds

$$\mu_E^U[A] = \mu_2^{EU}[EAE^{-1}] \quad . \tag{2.3}$$

- $\mu^U[\cdot]$ can be written as

$$\mu^U[A] \;=\; \lim_{h\to 0+} \frac{ln\|e^{hA}\|^U}{h} \quad . \tag{2.4}$$

- If $|\cdot|$ is an inner product norm $|\cdot|^2 =<\cdot,\cdot>$, then it holds

$$\mu^U[A] \;=\; \max_{z\in U, z\neq\, 0} \frac{< Az, z >}{|z|^2} \quad . \tag{2.5}$$

In the same analogous way estimates for the solutions of (2.1) in the invariant subspace $U(t)$ are obtained: Let $x(\cdot)$ and $\bar{x}(\cdot)$ be solutions of (2.1) in the invariant subspace $U(t)$. Denote the difference of the two solutions by $v(t) := x(t) - \bar{x}(t)$. Considering

$$m(t) := |v(t)| = |x(t) - \bar{x}(t)|$$

and taking into account that $v(t) = (x(t) - \bar{x}(t)) \in U(t)$ one obtains

$$\liminf_{h\to 0+} \frac{m(t+h) - m(t)}{h} \quad\leq\quad \mu^{U(t)}[W(t)] \cdot m(t)$$

and, hence, with $\quad L^U(t) \;=\; \int_0^t \mu^{U(\tau)}[W(\tau)]d\tau$

$$|v(t)| \quad\leq\quad e^{L^U(t)} \cdot |v(0)| \quad \forall t \geq 0.$$

It holds analogies to the theorem 1.1 and the corollary 1.2 with the only difference that the solutions must ly in the invariant subspace $U(t)$ and the logarithmic norm is taken with respect to this subspace.

Now, we ask , how do time-dependent transformations of variables influence the estimates? Let $E(\cdot)$ be a smooth matrix function and let $E(t)$ be nonsingular for all t. Aiming to estimate the difference of the transformed solutions $\tilde{v}(t) = E(t)v(t) = E(t)x(t) - E(t)\bar{x}(t)$ we look for a differential equation which is solved by them. Here we find

$$\begin{aligned}\tilde{v}'(t) = (Ev)'(t) \quad &= \quad E'(t)v(t) + E(t)v'(t) = E'(t)v(t) + E(t)W(t)v(t)\\ &= \quad (E'E^{-1} + EWE^{-1})(t)\tilde{v}(t)\\ &=: \quad \tilde{W}(t)\cdot\tilde{v}(t)\end{aligned} \tag{2.6}$$

for the difference of the solutions ,and, analogously,

$$\begin{aligned}\tilde{x}'(t) = (Ex)'(t) &= (E'E^{-1} + EWE^{-1})(t)\tilde{x}(t) + (Eq)(t) \\ &=: \tilde{W}(t) \cdot \tilde{x}(t) + \tilde{q}(t) \qquad (2.7)\end{aligned}$$

for the solutions itself. Further, we note that $\tilde{x}(t)$, $\tilde{v}(t) \in (EU)(t)$. Now, we obtain the estimate

$$\begin{aligned}|\tilde{v}(t)| &\leq \exp(\int_0^t \mu^{(EU)(s)}[(E'E^{-1} + EWE^{-1})(s)]ds) \cdot |\tilde{v}(0)| \\ &=: e^{\tilde{L}^{EU}(t)} \cdot |\tilde{v}(0)| \quad =: \quad \tilde{c}(t) \cdot |\tilde{v}(0)| \quad . \qquad (2.8)\end{aligned}$$

One may use the trivial estimate

$$\begin{aligned}|\tilde{v}(t)| &\leq \|E(t)\| \cdot |v(t)| \\ &\leq \|E(t)\| \cdot e^{L^U(t)} \cdot |v(0)| \\ &\leq \|E(t)\| \cdot e^{L^U(t)} \cdot \|E^{-1}(0)\| \cdot |\tilde{v}(0)| \\ &=: c(t) \cdot |\tilde{v}(0)|\end{aligned}$$

or, more accurate,

$$\begin{aligned}|\tilde{v}(t)| &\leq \|E(t)\|^{U(t)} \cdot e^{L^U(t)} \cdot \|E^{-1}(0)\|^{EU(0)} \cdot |\tilde{v}(0)| \\ &=: c^U(t) \cdot |\tilde{v}(0)|. \qquad (2.9)\end{aligned}$$

But, since there are used estimates of the form $|Ax| \leq ||A\| \cdot |x|$ or $|Ax| \leq ||A\|^U \cdot |x|$ for $x \in U$ one may loose information. **Remark:** For scalar equations it holds $c(t) = \tilde{c}(t)$. **Proof:** Let be $m = 1$, $E(t) = (\alpha(t))$ with $\alpha(t) \neq 0$,$W(t) = (\mu(t))$. Then one computes

$$c(t) = |\alpha(t)| \exp(\int_0^t \mu(\tau)d\tau) \cdot |\alpha(0)^{-1}| = \frac{\alpha(t)}{\alpha(0)} \cdot \exp(\int_0^t \mu(\tau)d\tau)$$

and

$$\begin{aligned}\tilde{c}(t) &= \exp(\int_0^t (\alpha'(\tau)\alpha^{-1}(\tau) + \alpha(\tau)\mu(\tau)\alpha^{-1}(\tau))d\tau) \\ &= \exp([\ln \alpha(\tau)]_0^t) \cdot \exp(\int_0^t \mu(\tau)d\tau) = \frac{\alpha(t)}{\alpha(0)} \cdot \exp(\int_0^t \mu(\tau)d\tau) = c(t)\end{aligned}$$

q.e.d.

Remark: For vector-valued equations ($m > 1$) we don't have $c(t) = \tilde{c}(t)$ in general. This can already be seen for constant coefficients W and constant scalings E , where we have

$$c(t) = \|E\| \cdot \|E^{-1}\| \cdot e^{\mu[W]t} \quad , \text{ and } \quad \tilde{c}(t) = e^{\mu[EWE^{-1}]t} \quad .$$

3 Logarithmic norms for index-1-DAEs

3.1 Linear index-1-DAEs

Consider the linear equation

$$A(t)x'(t) + B(t)x(t) = q(t), \quad t \in J \subset \mathbb{R}, \tag{3.1}$$

with continuous coefficients. Introduce the basic subspaces

$$\begin{aligned} N(t) &:= \ker A(t) \subset \mathbb{R}^m, \\ S(t) &:= \{z \in \mathbb{R}^m : B(t)z \in \operatorname{im} A(t)\} \subset \mathbb{R}^m, \end{aligned}$$

and assume $N(t)$ to be nontrivial as well as to vary smoothly with t Then, $A(t)$ has constant rank r. The null-space $N(t)$ determines what kind of functions we should accept for solutions of (2.1). To distinguish solution components which appear differentiated (differential components) and those components which do not (algebraic components) let us introduce a projector function $Q(t)$ onto $N(t)$, i.e.

$$Q(t)^2 = Q(t), \quad \operatorname{im} Q(t) = N(t), \quad t \in J.$$

Q shall be chosen such that it is as smooth as N . Further, let $P(t) := I - Q(t)$ denote the complementary projector. Then we may split a solution $x(t)$ and represent it as a sum of differential and algebraic components in the form

$$x(t) = (Px)(t) + (Qx)(t) \quad .$$

Now, the trivial identity $AQ = 0$ implies

$$Ax' = APx' = A(Px)' - AP'x$$

and, therefore, we use Ax' as an abbreviation of $A(Px)' - AP'x$ in the following. Thus, (3.1) may be rewritten as

$$A(Px)' + (B - AP')x = q, \tag{3.2}$$

which shows the function space

$$C_N^1(J, \mathbb{R}^m) := \{x \in C(J, \mathbb{R}^m) : Px \in C^1(J, \mathbb{R}^m)\}$$

to become the appropriate one for (3.1). The realization of both the expression Ax' and the space C_N^1 is independent of the special choice of the projector function (see e.g. [3]) . Obviously, $S(t)$ is the subspace in which the homogeneous equation solution proceeds. It is determined by the constraints and may be only continuous. Recall the condition

$$S(t) \oplus N(t) = \mathbb{R}^m, \quad t \in J, \tag{3.3}$$

to characterize the class of index-1 DAEs ([3]). Let us denote the special projector onto $N(t)$ which projects along $S(t)$ by

$$Q_{can}(t).$$

As the subspace $S(t)$ also the projector $Q_{can}(t)$ may be only continuous. Aiming to decouple the DAE (3.1) into some inherent dynamical part and an assignment for the derivative-free parts of the solution we will make use of these projectors.

The index-1 condition (3.3) implies the matrices

$$G(t) := (A + BQ)(t), \qquad \text{and} \qquad G_{can}(t) := (A + BQ_{can})(t) \tag{3.4}$$

to be nonsingular for all $t \in J$. Exploiting the relations (see [3])

$$G^{-1}A = P, \tag{3.5}$$

$$G^{-1}B = Q + G^{-1}BP, \tag{3.6}$$

$$QG^{-1}B = Q_{can} \tag{3.7}$$

we see that multiplying by G^{-1} or G_{can}^{-1} leads to a natural scaling of the dae. Multiplying (3.2) by $P_{can}G_{can}^{-1}$ and $Q_{can}G_{can}^{-1}$ we decouple this equation into the system

$$P_{can}x' + P_{can}G_{can}^{-1}BP_{can}x = P_{can}G_{can}^{-1}q \tag{3.8}$$

$$Q_{can}x = Q_{can}G_{can}^{-1}q. \tag{3.9}$$

Here, $P_{can}x'$ is used as an abbreviation for $P_{can}(Px)' - P_{can}P'x$. Since both projector functions P and P_{can} project along the same subspace N it holds

$$PP_{can} = P \qquad \text{and} \qquad P_{can}P = P_{can}. \tag{3.10}$$

Thus, multiplying the equation (3.8) by P leads to the equivalent equation

$$Px' + PG_{can}^{-1}BP_{can}Px = PG_{can}^{-1}q.$$

Using (3.9) we now compute

$$\begin{aligned} Px' &= (Px)' - P'x &= (Px)' - P'(P_{can}x + Q_{can}x) \\ &= (Px)' - P'(P_{can}Px + Q_{can}G_{can}^{-1}q) \end{aligned}$$

which leads us to an regular differential equation in $u := Px$, an so-called inherent regular differential equation:

$$u' + (-P'P_{can} + PG_{can}^{-1}BP_{can})u \;=\; (P + P'Q_{can})G_{can}^{-1}q. \tag{3.11}$$

If u solves (3.11) one easily computes that Qu solves the linear homogeneous ODE

$$(Qu)' - Q'Qu \;=\; 0,$$

Thus, we conclude from $u(t_0) \in \operatorname{im} P(t_0)$ or, equivalently, $(Qu)(t_0) = 0$ that for all t it holds $u(t) \in \operatorname{im} P(t)$, i.e. $\operatorname{im} P(t)$ is an invariant subspace for (3.11). Is u a solution of (3.11) with $u(t_0) \in \operatorname{im} P(t_0)$ then

$$x \;:=\; P_{can}u + Q_{can}G_{can}^{-1}q \tag{3.12}$$

is a solution of (3.1). Vice versa, each solution of (3.1) can be represented in the form (3.12) with $u(t_0) := P(t_0)x(t_0)$.

Supposed that the canonical projector function P_{can} is bounded the stability behavior of the solutions of the original DAE is determined by the solution properties of the inherent regular ODE (3.11) in the invariant subspace $\operatorname{im} P(t)$.
In general the supposition that P_{can} is bounded may be rather restrictive and should be replaced by the condition that P_{can} behaves moderately.

Remark 3.1 *Starting the decoupling of the DAE (3.1) by multiplying by PG^{-1} and QG^{-1} is just another technique to obtain the same inherent regular equation (3.11) .*

Proof: (3.1) is equivalent to

$$Px' + PG^{-1}BPx \;=\; PG^{-1}q \tag{3.13}$$

$$Qx + Q_{can}Px \;=\; QG^{-1}q. \tag{3.14}$$

Using (3.14) we obtain

$$\begin{aligned} Px' &= (Px)' - P'x = (Px)' - P'(Px + Qx) \\ &= (Px)' - P'(Px - Q_{can}Px + QG^{-1}q) \\ &= (Px)' - P'(P_{can}x + QG^{-1}q) \quad . \end{aligned}$$

This leads to the following regular differential equation in $u := Px$:

$$u' + (-P'P_{can} + PG^{-1}BP)u \;=\; (P + P'Q)G^{-1}q. \tag{3.15}$$

(3.15) coincides with (3.11). To show this we will check now, that the coefficients coincide. First, we look at the right-hand sides. Here we see:

$$(P + P'Q)G^{-1} = (P + P'Q_{can})G_{can}^{-1} \qquad \text{, since}$$

$$\begin{aligned} (P + P'Q)G^{-1}G_{can} &= (P + P'Q)G^{-1}(A + BQ_{can}) = (P + P'Q)(G^{-1}A + G^{-1}BQ_{can}) \\ &= (P + P'Q)(P + (Q + G^{-1}BP)Q_{can}) = (P + P'Q)(P + Q_{can}) \\ &= P + P'Q_{can} \quad . \end{aligned}$$

Next, we look at the coefficient matrices. Here the first term $-P'P_{can}$ is obviously the same in both matrices. We will see that also the second terms coincide. First, we note that $\quad PG^{-1}BP = PG^{-1}BP_{can} \quad$. This equality follows since
$QQ_{can} = Q_{can}$, hence $G^{-1}BQ_{can} = G^{-1}BQQ_{can} = QQ_{can} = Q_{can}$ and ,hence
$PG^{-1}BP - PG^{-1}BP_{can} = PG^{-1}BQ_{can} - PG^{-1}BQ = PQ_{can} - PQ = 0 \quad$.
Then, we compute

$$\begin{aligned} & PG^{-1}BP - PG_{can}^{-1}BP_{can} \;=\; PG^{-1}BP_{can} - PG_{can}^{-1}BP_{can} \\ =\; & G^{-1}(AG^{-1}BP_{can} - AG_{can}^{-1}BP_{can}) \\ =\; & G^{-1}((G - BQ)G^{-1}BP_{can} - (G_{can} - BQ_{can})G_{can}^{-1}BP_{can}) \\ =\; & G^{-1}(BP_{can} - B\underbrace{QG^{-1}B}_{Q_{can}}P_{can} - BP_{can} + B\underbrace{Q_{can}G_{can}^{-1}B}_{Q_{can}}P_{can} \quad = \quad 0 \quad . \end{aligned}$$

Summarizing we have

$$PG^{-1}BP = PG^{-1}BP_{can} = PG_{can}^{-1}BP_{can} \quad . \tag{3.16}$$

q.e.d.

As P is not uniquely determined also the inherent regular differential equation is not unique. Let $\tilde{P}$ be some other smooth projector function along N. Then we have with

$$\tilde{u}' + (-\tilde{P}'P_{can} + \tilde{P}G_{can}^{-1}BP_{can})\tilde{u} \;=\; (\tilde{P} + \tilde{P}'Q_{can})G_{can}^{-1}q \tag{3.17}$$

another inherent regular differential equation in $\tilde{u} := \tilde{P}x$ with $\operatorname{im}\tilde{P}$ as an invariant subspace. Again we obtain by

$$x \;:=\; P_{can}\tilde{u} + Q_{can}G_{can}^{-1}q \tag{3.18}$$

a solution of (3.2) if $\tilde{u}$ is a solution of (3.17) with $\tilde{u}(t_0) \in \operatorname{im}\tilde{P}(t_0)$.

If the subspace $S(t)$ depends smoothly on t also the canonical Projector $P_{can} = I - Q_{can}$ may be chosen as a smooth projector function along N . We then obtain

$$u_{can}' + (-P_{can}'P_{can} + P_{can}G_{can}^{-1}BP_{can})u_{can} \;=\; (P_{can} + P_{can}'Q_{can})G_{can}^{-1}q \tag{3.19}$$

as inherent regular differential equation in $u_{can} := P_{can}x$ with $S(t) = \operatorname{im}P_{can}(t)$ as an invariant subspace. Solutions of (3.2) are composed by

$$x \;:=\; u_{can} + Q_{can}G_{can}^{-1}q \tag{3.20}$$

if u_{can} is a solution of (3.19) with $u_{can}(t_0) \in S(t_0) = \operatorname{im}P_{can}(t_0)$.

Aiming at estimates for the solutions of (3.1)we use the composition of the solutions (3.12) and apply the concept of logarithmic matrix norms for ODEs with invariant subspaces to the inherent regular ODE (3.11). This leads us to the following theorem:

Theorem 3.2 *Let x and $\bar{x}$ be both solutions of (3.1). Let denote*

$$L(t) \;=\; \int_0^t \mu^{\operatorname{im}P(\tau)}[-M(\tau)]d\tau \quad \text{, where} \quad M(t) = (-P'P_{can} + PG_{can}^{-1}BP_{can})(t)$$

is the coefficient matrix in the inherent regular ODE . Further, let q_P denote the right-hand side of the inherent regular ODE (3.11) : $\quad q_P = (P + P'Q_{can})G_{can}^{-1}q$. Then it holds for all $t \geq 0$:

$$\begin{aligned} |x(t)| \;\leq\; & \|P_{can}(t)\|^{\operatorname{im}P(t)} \cdot \left(e^{L(t)} \cdot |(Px)(0)| + e^{L(t)}\int_0^t e^{-L(s)}|q_P(s)|ds\right) \\ & +|(Q_{can}G_{can}^{-1}q)(t)|, \end{aligned} \tag{3.21}$$

and

$$|(x-\bar{x})(t)| \leq \|P_{can}(t)\|^{\operatorname{im}P(t)} \cdot e^{L(t)} \cdot |(P(x-\bar{x}))(0)|. \tag{3.22}$$

Remark: Under the condition that $S(t)$ is smooth we may choose $P := P_{can}$ which leads directly to $P_{can}x = u_{can}$ (see (3.20)). In relation to this we see that

$$\|P_{can}\|^S = \|P_{can}\|^{\mathrm{im}\,P_{can}} = \max_{z\in \mathrm{im}\,P_{can}, z\neq 0} \frac{|P_{can}z|}{|z|} = \max_{w: P_{can}w\neq 0} \frac{|P_{can}P_{can}w|}{|P_{can}w|} = 1 \quad .$$

With $\quad L_{can}(t) = \int_0^t \mu^{S(\tau)}[-M_{can}(\tau)]d\tau \quad$ where $\quad M_{can} = -P'_{can}P_{can} + P_{can}G_{can}^{-1}BP_{can}$ we obtain as a special case of (3.21) and (3.22)

$$\begin{aligned} |x(t)| &\leq e^{L_{can}(t)} \cdot |(P_{can}x)(0)| + e^{L_{can}(t)} \int_0^t e^{-L_{can}(s)}|q_{P_{can}}(s)|ds + |(Q_{can}G_{can}^{-1}q)(t)|, \\ |(x-\bar{x})(t)| &\leq e^{L_{can}(t)} \cdot |(P_{can}(x-\bar{x}))(0)|. \end{aligned} \tag{3.23}$$

Now, the question arises, how do these estimates depend on the chosen projector P ? First, note that only the first term $P_{can}x = P_{can}Px = P_{can}u$ of (3.12) causes estimates that may depend on the special choice of the projector P. Let us assume that $P$and $\tilde{P}$ are both smooth projector functions along N. Choosing $u := Px$ resp. $\tilde{u} := \tilde{P}x$ as differential components of the solution x we obtain (3.11) resp. (3.17) as the inherent regular ODE. Now, we will show that these underlying inherent regular ODEs represent transformations of each other.

Theorem 3.3 *Let $P$and $\tilde{P}$ are both smooth projector functions along N. Let denote $Q = I - P$ and $\tilde{Q} = I - \tilde{P}$ the corresponding complementary projectors. Then , $E = \tilde{P} + Q$ is a smooth matrix function and $E(t)$ is nonsingular for all t . It is $E^{-1} = P + \tilde{Q}$ and $E \cdot P = \tilde{P}$. It holds: If u is a solution of (3.11) with $u(t) \in \mathrm{im}\,P(t)$ then $\tilde{u} := E \cdot u$ is a solution of (3.17) with $\tilde{u}(t) \in \mathrm{im}\,\tilde{P}$.*

Proof: Let u be a solution of (3.11) with $u(t) \in \mathrm{im}\,P(t)$ and $\tilde{u} := E \cdot u$. Then it holds $\tilde{u}(t) \in E(t)\mathrm{im}\,P(t) = \mathrm{im}\,\tilde{P}(t)$ and

$$\begin{aligned} \tilde{u}' = (Eu)' &= E'u + Eu' = E'u + E(-Mu + (P + P'Q_{can})G_{can}^{-1}q) \\ &= (E'E^{-1} - EME^{-1})(Eu) + E(P + P'Q_{can})G_{can}^{-1}q \quad . \end{aligned}$$

First, we deal with the inhomogeneous term. It is:

$$\begin{aligned} E(P + P'Q_{can}) &= (\tilde{P} + Q)(P + P'Q_{can}) = \tilde{P} + \underbrace{\tilde{P}P'}_{\tilde{P}' - \tilde{P}'P} Q_{can} + \underbrace{QP'}_{-Q'P} Q_{can} \\ &= \tilde{P} + \tilde{P}'Q_{can} \end{aligned}$$

since $PQ_{can} = 0$. Next, we deal with the coefficient matrix. Since $Eu \in \mathrm{im}\tilde{P}$ for $u \in \mathrm{im}\,P$ we are interested in the values of this coefficient matrix applied to elements of $\mathrm{im}\,\tilde{P}$ only. Here we compute:

$$\begin{aligned}
(E'E^{-1} - EME^{-1})\tilde{P} &= (E' - EM)(P + \tilde{Q})\tilde{P} = E'P - EMP \\
&= (\tilde{P}' + Q')P - (\tilde{P} + Q)(-P'P_{can} + PG_{can}^{-1}BP_{can})P \\
&= \tilde{P}'P + Q'P + \underbrace{\tilde{P}P'}_{\tilde{P}' - \tilde{P}'P} P_{can} + \underbrace{QP'}_{-Q'P} P_{can} - \tilde{P}G_{can}^{-1}BP_{can} \\
&= \tilde{P}'P_{can} - \tilde{P}G_{can}^{-1}BP_{can} \quad = \quad -\tilde{M} = -\tilde{M}\tilde{P} \quad .
\end{aligned}$$

Summarizing it follows that $\tilde{u}$ solves (3.17). q.e.d.

Corollary 3.4 *Under the suppositions of theorem 3.3 it holds*

$$\mu^{\mathrm{im}\,\tilde{P}(t)}[-\tilde{M}(t)] = \mu^{E(t)\mathrm{im}\,P(t)}[(E'E^{-1} + E(-M)E^{-1})(t)] \quad .$$

3.2 Relation to the logarithmic norm for matrix pencils

There is another idea to extend the usual concepts of induced matrix norm and logarithmic matrix norm by Higueras and Garcia-Celayeta [4] . They define a logarithmic norm for matrix pencils and use this to study the growth of the solutions of linear time-dependent DAEs. We start quoting their definitions, some remarks and main theorems.

Definition 3.5 *Let $A, B \in L(\mathbb{R}^m)$, let $V \subseteq \mathbb{R}^m$ a subspace such that $V \cap \ker A = \{0\}$. Then we call V an admissible subspace for $\|A, \cdot\|$ and define*

$$\|A, B\|^V = \max_{v \in V, |Av| \neq 0} \frac{|Bv|}{|Av|} \tag{3.24}$$

and

$$\mu^V[A, B] = \lim_{h \to 0^+} \frac{\|A, A - hB\|^V - 1}{h} \quad . \tag{3.25}$$

Remarks:

- For $A = I$ and $V = \mathbb{R}^m$ it is $\mu^{\mathbb{R}^m}[I, B] = \mu[-B]$.
- If $|\cdot|$ is an inner product norm, then

$$\mu^V[A, B] = \max_{x \in V, Ax \neq 0} \frac{< Ax, -Bx >}{< Ax, Ax >} \quad .$$

- If E is nonsingular and $|x|_E = |Ex|_2$ then

$$\mu_E^V[A,B] = \mu_2^V[EA, EB] \quad \text{and} \quad \mu^V[A,B] = \mu^{EV}[AE^{-1}, BE^{-1}] \quad .$$

It holds the following theorem concerning the growth of the solutions of the linear homogeneous DAE

$$A(t)x'(t) + B(t)x(t) = 0 \qquad : \tag{3.26}$$

Theorem 3.6 *Let x be a solution of (3.26) such that $x(t) \in V(t)$ where $V(t) \subseteq \mathbb{R}^m$ is an admissible subspace for $\|A(t), \cdot\|$. Then it holds*

$$|(Ax)(t)| \le e^{\int_0^t \mu^{V(\tau)}[A(\tau), B(\tau) - A'(\tau)]d\tau} \cdot |(Ax)(0)| \qquad \forall t \ge 0 \quad . \tag{3.27}$$

Corollary 3.7 *Let (3.26) have index 1. Then the solutions of (3.26) lie in $S(t)$ and $S(t)$ is an admissible subspace for $\|A(t), \cdot\|$. Hence, it holds the estimate (3.27) with $V(t) := S(t)$:*

$$\begin{aligned} |(Ax)(t)| \ &\le \ e^{L_A(t)} \cdot |(Ax)(0)| \qquad \forall t \ge 0 \quad , \\ &\quad \textit{where} \quad L_A(t) = \int_0^t \mu^{S(\tau)}[A(\tau), B(\tau) - A'(\tau)]d\tau \, . \end{aligned} \tag{3.28}$$

We now want to relate these results to those of the previous chapter. Therefore, let x be a solution of the linear homogeneous DAE (3.26) and let (3.26) have index 1. Then it holds $\quad x = P_{can}x = P_{can}u \quad$, where u solves the homogeneous inherent regular ODE $u' = -Mu \quad , u(0) := P(0)x(0) \quad$ (see (3.9),(3.11) with $q = 0$). This yields the estimate

$$\begin{aligned} |(Px)(t)| \ &\le \ e^{L(t)} \cdot |(Px)(0)| \qquad \forall t \ge 0 \quad , \\ &\quad \text{where} \quad L(t) = \int_0^t \mu^{\mathrm{im}\, P(\tau)}[-M(\tau)]d\tau \, . \end{aligned} \tag{3.29}$$

The estimate (3.29) concerns Px , while (3.28) concerns Ax. Note that it holds $A = GP$, G is a smooth matrix function and $G(t)$ is nonsingular for all t . We may now start from the inherent regular ODE (3.11) in $u := Px$ and look for the corresponding ODE for the transformed variables $\hat{u} := Gu = GPx = Ax$. This results in

$$\hat{u}' = (G'G^{-1} + G(-M)G^{-1})u \ := \ -\hat{M}u \quad , \quad \hat{u}(0) := (GPx)(0) = (Ax)(0) \tag{3.30}$$

and gives estimates

$$|(Ax)(t)| \leq e^{\hat{L}(t)} \cdot |(Ax)(0)| \qquad \forall t \geq 0 \quad , \tag{3.31}$$
$$\text{where} \quad \hat{L}(t) = \int_0^t \mu^{\operatorname{im} A(\tau)}[(G'G^{-1} + G(-M)G^{-1})(\tau)]d\tau \; .$$

Now, we show that this estimate coincides with the estimate (3.28) obtained by Higueras and Garcia-Celayeta [4] with the help of their logarithmic norm for matrix pencils:

Theorem 3.8 *Let the linear time-dependent DAE (3.1) have index 1. Then it holds*

$$\mu^{S(t)}[A(t), B(t) - A'(t)] = \mu^{\operatorname{im} A(t)}[(G'G^{-1} + G(-M)G^{-1})(t)] \quad \forall t$$

and ,hence , $\quad L_A(t) = \hat{L}(t) \quad \forall t \geq 0$.

Proof: By definition it is

$$\mu^S[A, B - A'] = \liminf_{h \to 0^+} \frac{\|A, A - h(B - A')\|^S - 1}{h} \quad \text{where} \quad \|A, C\|^S = \max_{v \in S, Av \neq 0} \frac{|Cv|}{|Av|}$$

and

$$\mu^{\operatorname{im} A}[-\hat{M}] = \liminf_{h \to 0^+} \frac{\|I + h(-\hat{M})\|^{\operatorname{im} A} - 1}{h} \quad \text{where} \quad \|W\|^U = \max_{u \in U, u \neq 0} \frac{|Wu|}{|u|}$$

It remains to show that $\quad \|A, A - h(B - A')\|^S = \|I - h\hat{M}\|^{\operatorname{im} A} \quad$,i.e.

$$\max_{v \in S, Av \neq 0} \frac{|(A - h(B - A'))v|}{|Av|} = \max_{u \in \operatorname{im} A, u \neq 0} \frac{|(I - h\hat{M})u|}{|u|} \quad .$$

Since

$$\max_{u \in \operatorname{im} A, u \neq 0} \frac{|(I - h\hat{M})u|}{|u|} = \max_{v, Av \neq 0} \frac{|(I - h\hat{M})Av|}{|Av|} = \max_{v = P_{can}v, Av \neq 0} \frac{|(I - h\hat{M})Av|}{|Av|}$$

it remains to show that $-\hat{M}Av = (A' - B)v \quad \forall v \in S$,i.e. $-\hat{M}AP_{can} = (A' - B)P_{can}$. But this can be seen by:

$$\begin{aligned}
\hat{M}AP_{can} &= \hat{M}A = (-G'G^{-1} + GMG^{-1})A = -G'P + GMP \\
&= -(A + BQ)'P + (A + BQ)(-P'P_{can} + PG_{can}^{-1}BP_{can}) \\
&= -(A + BQ)'P - (A + BQ)P'P_{can} + AG_{can}^{-1}BP_{can} \\
&= -(A + BQ)'P - (A' - (A + BQ)'P)P_{can} + (G_{can} - BQ_{can})G_{can}^{-1}BP_{can} \\
&= -A'PP_{can} + BP_{can} - BQ_{can}P_{can} \quad = \quad -(A' - B)P_{can} \quad .
\end{aligned}$$

q.e.d.

Finally, we consider the special case that the DAE (3.1) is already scaled such that $A = P$ is a projector function (this includes semi-explicit DAEs where $A = P = \begin{pmatrix} I & \\ & 0 \end{pmatrix}$) . Here, (3.28) as well (3.29) supply estimates of Px. Corresponding to that we will show that then the constants in (3.28) and (3.29) coincide, i.e. the value of the logarithmic norm for the pencil (P, B) with respect to the subspace $S(t)$ coincides with the value of the logarithmic norm for the inherent regular equation with respect to the invariant subspace $\operatorname{im} P$.

Theorem 3.9 *Let the linear time-dependent DAE (3.1) have index 1 and let A be a smooth projector function, i.e. $A = P$. Then it holds*

$$\mu^{S(t)}[P(t), B(t) - P'(t)] = \mu^{\operatorname{im} P(t)}[-M(t)] \quad \forall t$$

Proof: Following the steps in the proof of theorem3.8 it remains to show that $-MP_{can} = (P' - B)P_{can}$. This can be seen by

$$\begin{aligned} MP_{can} &= -P'P_{can} + PG_{can}^{-1}BP_{can} \\ &= -P'P_{can} + (G_{can} - BQ_{can})G_{can}^{-1}BP_{can} \\ &= -P'P_{can} + BP_{can} \quad = \quad -(P' - B)P_{can} \end{aligned}$$

q.e.d.

Remark: Applying this result to an semi-explicit index-1 DAE

$$Ax' + Bx := \begin{pmatrix} I & \\ & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in I\!R^{m_1} \times I\!R^{m_2}$$

we obtain

$$N = \{z : z_1 = 0\} \quad , \quad S = \{z : B_{21}z_1 + B_{22}z_2 = 0\} = \{z : z_2 = -B_{22}^{-1}B_{21}z_1\} \, ,$$

$$M = \begin{pmatrix} B_{11} - B_{12}B_{22}^{-1}B_{21} & 0 \\ 0 & 0 \end{pmatrix} \text{ for } P := A = \begin{pmatrix} I & \\ & 0 \end{pmatrix} \quad \text{while} \quad P_{can} = \begin{pmatrix} I & 0 \\ -B_{22}^{-1}B_{21} & 0 \end{pmatrix}$$

and, hence

$$\mu^S[A, -B] = \mu^{R^{m_1} \times \{0\}^{m_2}}[-M] = \mu[-(B_{11} - B_{12}B_{22}^{-1}B_{21})] \quad .$$

3.3 Examples

3.3.1 Example 1

We consider the linear DAE (3.1) with the coefficients

$$A(t) = \begin{pmatrix} 1 & 0 \\ \eta t & 0 \end{pmatrix} \quad , \quad B(t) = I \quad .$$

The solutions of the homogeneous DAE can be represented in the form

$$x(t) = \begin{pmatrix} 1 \\ \eta t \end{pmatrix} e^{-t} x_1(0) \quad .$$

For the subspaces $N(t)$ and $S(t)$ we obtain

$$N(t) = \ker A(t) = \operatorname{span}\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\} \quad \text{and} \quad S(t) = \operatorname{im} A(t) = \operatorname{span}\{\begin{pmatrix} 1 \\ \eta t \end{pmatrix}\} .$$

In this example A is itself a projector function onto im $A = S$ such that $A = P_{can}$ and $G_{can} = A + BQ_{can} = P_{can} + Q_{can} = I$.

Now, let x be a solution of the linear homogeneous DAE (3.26).We want to see what the presented estimates for the growth of solutions give here . As mentioned there is a variety of possibilities to split the solution in differential and algebraic components. Choosing the constant orthogonal projector $\tilde{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ along N would yield a splitting

$$x(t) = \tilde{P}x(t) + \tilde{Q}x(t) = \begin{pmatrix} x_1(t) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2(t) \end{pmatrix} \quad .$$

Choosing the canonical projector $P_{can}(t) = A(t)$ yield

$$x(t) = P_{can}(t)x(t) + Q_{can}(t)x(t) = \begin{pmatrix} x_1(t) \\ \eta t x_1(t) \end{pmatrix} + \begin{pmatrix} 0 \\ -\eta t x_1(t) + x_2(t) \end{pmatrix} \quad .$$

First we note that because of (3.9) for the solution of the homogeneous DAE $\quad Q_{can}x = 0 \quad$ is fulfilled such that

$$x \quad = \quad P_{can}x + Q_{can}x \quad = \quad P_{can}x \quad .$$

Based on this different splittings there are different possibilities to estimate the growth of $x = P_{can}x$:

- Choosing an inherent regular ODE (3.19) in $u_{can} = P_{can}x$ we obtain the coefficient matrix

$$\begin{aligned} M_{can}(t) &= (-P'_{can}P_{can} + P_{can}G_{can}^{-1}BP_{can})(t) \quad = \quad (-A'A + AIA)(t) \\ &= -\begin{pmatrix} 0 & 0 \\ \eta & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ \eta t & 0 \end{pmatrix} \quad = \quad \begin{pmatrix} 1 & 0 \\ \eta t - \eta & 0 \end{pmatrix} \end{aligned}$$

and compute

$$\begin{aligned} \mu_2^{S(t)}[-M_{can}(t)] &= \max_{x\in S(t), x\neq 0} \frac{< -M_{can}(t)x, x >_2}{< x, x >_2} \quad = \quad \frac{< -M_{can}(t)\begin{pmatrix} 1 \\ \eta t \end{pmatrix}, \begin{pmatrix} 1 \\ \eta t \end{pmatrix} >}{< \begin{pmatrix} 1 \\ \eta t \end{pmatrix}, \begin{pmatrix} 1 \\ \eta t \end{pmatrix} >} \\ &= \frac{< -\begin{pmatrix} 1 \\ \eta t - \eta \end{pmatrix}, \begin{pmatrix} 1 \\ \eta t \end{pmatrix} >}{1 + (\eta t)^2} \quad = \quad -1 + \frac{\eta^2 t}{1 + (\eta t)^2} \end{aligned}$$

By theorem 3.9 we also have

$$\begin{aligned} \mu_2^{S(t)}[A(t), B(t) - A'(t)] &= \mu_2^{S(t)}[P_{can}(t), I - P'_{can}(t)] \quad = \quad \mu_2^{S(t)}[-M_{can}(t)] \\ &= -1 + \frac{\eta^2 t}{1 + (\eta t)^2} \quad . \end{aligned}$$

With $L_{can}(t) = \int_0^t \mu^{S(\tau)}[-M_{can}(\tau)]d\tau = \int_0^t (-1 + \frac{\eta^2 \tau}{1+(\eta\tau)^2})d\tau$ and (3.23) we have

$$|x(t)| \leq e^{L_{can}(t)} \cdot |(P_{can}x)(0)| \; = \; e^{L_{can}(t)} \cdot |x(0)|$$

- Choosing the constant orthogonal projector $\tilde{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ along N and an inherent regular ODE (3.17) in $\tilde{u} = \tilde{P}x$ we obtain the coefficient matrix

$$\begin{aligned} \tilde{M}(t) &= (-\tilde{P}'P_{can} + \tilde{P}G_{can}^{-1}BP_{can})(t) \quad = \quad (\tilde{P}G_{can}^{-1}BP_{can})(t) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \eta t & 0 \end{pmatrix} \quad = \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and compute

$$\begin{aligned} \mu_2^{\operatorname{im}\tilde{P}}[-\tilde{M}] &= \max_{x\in \operatorname{im}\tilde{P}, x\neq 0} \frac{< -\tilde{M}x, x >_2}{< x, x >_2} \quad = \quad \frac{< -\tilde{M}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} >}{< \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} >} \\ &= -1 \quad . \end{aligned}$$

Because of $\tilde{L}(t) = \int_0^t (-1) d\tau = -t$ we obtain

$$|\tilde{P}x(t)| \leq e^{-t} |\tilde{P}x(0)| \quad \forall t \geq 0$$

and

$$\begin{aligned} |x(t)| = |P_{can}(t)\tilde{P}x(t)| &\leq \|P_{can}(t)\|^{\mathrm{im}\,\tilde{P}} e^{-t} \cdot |\tilde{P}x(0)| \\ &\leq \|P_{can}(t)\|^{\mathrm{im}\,\tilde{P}} e^{-t} \cdot \|\tilde{P}\|^{\mathrm{im}\,P_{can}(0)} |x(0)| \quad . \end{aligned}$$

Because of $P_{can}(0) = \tilde{P}$ it is $\|\tilde{P}\|^{\mathrm{im}\,P_{can}(0)} = \|\tilde{P}\|^{\mathrm{im}\,\tilde{P}} = 1$. Further, we compute:

$$\|P_{can}(t)\|^{\mathrm{im}\,\tilde{P}} = \max_{x \in \mathrm{im}\,\tilde{P}, x \neq 0} \frac{P_{can}(t)x|}{|x|} = \frac{P_{can}(t)e_1|}{|e_1|} = \frac{|\begin{pmatrix} 1 \\ \eta t \end{pmatrix}|}{1} = \sqrt{1 + (\eta t)^2} \quad .$$

Hence, it follows

$$|x(t)| = |P_{can}(t)x(t)| \leq \sqrt{1 + (\eta t)^2}\, e^{-t}\, |P_{can}(0)x(0)| = \sqrt{1 + (\eta t)^2}\, e^{-t}\, |x(0)| \quad .$$

Note that we end with the same result if we apply the estimates based on the logarithmic norm for matrix pencils to the DAE scaled by $\tilde{G}^{-1} = (A + B\tilde{Q})^{-1} = (P_{can} + \tilde{Q})^{-1}$. We then obtain coefficients

$$\tilde{A} = \tilde{G}^{-1}A = \tilde{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \tilde{B} = \tilde{G}^{-1}B = \tilde{G}^{-1} = \tilde{P} + Q_{can} = \begin{pmatrix} 1 & 0 \\ -\eta t & 1 \end{pmatrix} \quad .$$

Since a scaling of the DAE does not change the solutions, the subspaces $N = \tilde{N}$ and $S = \tilde{S}$ do not change. Therefore, computing

$$\begin{aligned} \mu_2^{S(t)}[\tilde{A}, \tilde{B}(t)] &= \max_{x \in S(t), x \neq 0} \frac{< \tilde{A}x, -\tilde{B}x >}{< \tilde{A}x, \tilde{A}x >} = \frac{< \tilde{A}\begin{pmatrix} 1 \\ \eta t \end{pmatrix}, -\tilde{B}\begin{pmatrix} 1 \\ \eta t \end{pmatrix} >}{< \tilde{A}\begin{pmatrix} 1 \\ \eta t \end{pmatrix}, \tilde{A}\begin{pmatrix} 1 \\ \eta t \end{pmatrix} >} \\ &= \frac{< \begin{pmatrix} 1 \\ 0 \end{pmatrix}, -\begin{pmatrix} 1 & 0 \\ -\eta t & 1 \end{pmatrix} >}{< \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} >} = -1 \end{aligned}$$

we again obtain the estimate

$$\begin{aligned} |(\tilde{A}x)(t)| = |(\tilde{P}x)(t)| &\leq e^{-t} |(\tilde{P}x)(0)| \quad \forall t \geq 0 \quad , \\ |(Ax)(t)| = |(\tilde{G}\tilde{A}x)(t)| &\leq \|\tilde{G}(t)\|^{\mathrm{im}\,\tilde{P}} \cdot e^{-t} |(\tilde{P}x)(0)| \\ &= \sqrt{1 + (\eta t)^2}\, e^{-t}\, |x(0)| \quad . \end{aligned}$$

Since $\dim \operatorname{im} \tilde{P} = \dim \operatorname{im} P_{can}(t) = 1$ we meet the same nice property that all this estimates coincide like for scalar ODEs. Indeed, we compute

$$\sqrt{1+(\eta t)^2}\, e^{-t} = e^{L_{can}(t)}$$

since

$$\begin{aligned} L_{can}(t) &= \int_0^t \mu^{S(\tau)}[-M_{can}(\tau)]d\tau &&= \int_0^t (-1+\frac{\eta^2\tau}{1+(\eta\tau)^2})d\tau \\ &= -t+\int_0^t \frac{\eta^2\tau}{1+(\eta\tau)^2})d\tau &&= -t+\int_0^t (ln\sqrt{1+\eta^2\tau})'d\tau \\ &= -t+ln\sqrt{1+\eta^2 t} \quad . \end{aligned}$$

3.4 Nonlinear index-1 DAEs

We consider nonlinear DAEs

$$f(x'(t),x(t),t)=0 \quad , \tag{3.32}$$

where f is defined on $I\!R^m \times \mathcal{D} \times \mathcal{J}$ with $\mathcal{D} \subseteq I\!R^m$, $\mathcal{J} \subseteq I\!R$. f is supposed to be at least continuous and to be continuously differentiable with respect to the first and second argument. We will use the concept of tractability with index 1 introduced by Griepentrog/März [3]. Using the notations:

$$\begin{aligned} A(y,x,t) &:= f'_y(y,x,t) \quad , \quad B(y,x,t) := f'_x(y,x,t) \quad , \\ S(y,x,t) &:= \{z \in I\!R^m : B(y,x,t)z \in \operatorname{im} A(y,x,t)\} \quad \forall\, (y,x,t) \in I\!R^m \times \mathcal{D} \times \mathcal{J} \end{aligned}$$

we define

Definition 3.10 *The DAE (3.32) has (tractability -) index 1 in* $I\!R^m \times \mathcal{D} \times \mathcal{J}$ *iff*

- $\ker A(y,x,t)$ *does not depend on* y *and* x*, and*

  $$N(t) := \ker A(y,x,t)$$

  is smooth, and

- $N(t) \cap S(y,x,t) = \{0\} \quad \forall\, (y,x,t) \in I\!R^m \times \mathcal{D} \times \mathcal{J}$.

- *for a smooth projector function* $Q(\cdot)$ *onto* $N(\cdot)$ *the inverse of the nonsingular matrix*

$$G(y,x,t) := A(y,x,t) + B(y,x,t)Q(t)$$

is uniformly bounded in $I\!R^m \times \mathcal{D} \times \mathcal{J}$.

As for linear index-1 DAEs we again introduce a smooth projector function $Q(\cdot)$ onto $N(\cdot)$ and denote the complementary projector function by $P(\cdot) := I - Q(\cdot)$. Then, it holds

$$\begin{aligned} f(y,x,t) &= f(P(t)y,x,t) \quad \text{and} \\ A(y,x,t) &= A(P(t)y,x,t) \quad \text{and} \\ B(y,x,t) &= B(P(t)y,x,t) \quad \forall\,(y,x,t) \in I\!R^m \times \mathcal{D} \times \mathcal{J} \end{aligned}$$

and we will consider (3.32) as an abbreviation of the more accurate formulation

$$f((Px)'(t) - P'(t)x(t), x(t), t) = 0 \quad . \tag{3.33}$$

Again ,we are looking for solutions in the function space $C^1_N(\mathcal{J}, I\!R^m)$.
Aiming to estimate the error-growth or the growth of differences of solutions to each other we again try to find an inherent regular (nonlinear) ODE for the dynamical part of the solutions and an assignment for the derivative-free parts . Though there is no such decoupling for general nonlinear higher index DAEs (since this would involve differentiation steps to find hidden constraints) this is well-understood for nonlinear DAEs with (tractability-) index 1. We will give a short description of of this nonlinear decoupling. For details we refer to [3].

Lemma 3.11 *Let (3.32) have (tractability -) index 1 in* $I\!R^m \times \mathcal{D} \times \mathcal{J}$ *and let* $f(y_0, x_0, t_0) = 0$ *be fulfilled for a consistent initialization* $(y_0, x_0, t_0) \in I\!R^m \times \mathcal{D} \times \mathcal{J}$. *Then the nonlinear equation*

$$f(w, u + Q(t)w, t) = 0 \tag{3.34}$$

is locally uniquely solvable for $w = w(u,t)$ *with* $w(P(t_0)x_0, t_0) = P(t_0)y_0 + Q(t_0)x_0 := w_0$.

Proof: The corresponding Jacobian of (3.34) in $(w_0, u_0, t_0) := (P(t_0)y_0 + Q(t_0)x_0, P(t_0)x_0, t_0)$ is $A(y_0, x_0, t_0) + B(y_0, x_0, t_0)Q(t_0) = G(y_0, x_0, t_0)$ which is nonsingular since (3.32) has index 1 and the assertion follows immediately by the Implicit Function Theorem. q.e.d.

Now, for $u := Px$ and $x = u + Qw$ we obtain

$$f(x', x, t) = f((Px)' - P'x, x, t) = f(u' - P'(u + Qw), u + Qw, t) = 0$$

and then from Lemma3.11

$$u' - P'(u + Qw) = Pw \quad , \quad w = w(u, t) \quad \text{solution of (3.34).} \tag{3.35}$$

It is possible to show that (3.35) has $\operatorname{im} P(t)$ as an invariant subspace.Thus we have derived a nonlinear inherent regular ODE for $u := Px$ and a nonlinear equation which gives us $Qx = Qw(u, t)$. This is sometimes called a state-space-form of the DAE (3.32). We formulate the results in the following theorem:

Theorem 3.12 *Let (3.33) have (tractability -) index 1 in* $I\!R^m \times \mathcal{D} \times \mathcal{J}$ *.Then it holds:*

- *If* $x(\cdot) \in C^1_N(\mathcal{J}, I\!R^m)$ *is a solution of (3.33) then* $u(\cdot) := (Px)(\cdot)$ *solves the ODE*

$$\begin{aligned} u'(t) &= P'(t)u(t) + P(t)(I + P'(t))w(u(t), t) \quad , \\ u(t_0) &= P(t_0)x(t_0) \end{aligned} \tag{3.36}$$

  where $w = w(u, t)$ *is uniquely defined by* $f(w, u + Q(t)w, t) = 0$.

- *If* $u(\cdot)$ *with* $u(t_0) \in \operatorname{im} P(t_0)$ *solves the inherent regular ODE (3.36) then*

$$x(t) = u(t) + Q(t)w(u(t), t) \tag{3.37}$$

  is a solution of (3.33) and, furthermore, $u(t) := (Px)(t)$ *and*
  $w(u(t), t) = P(t)(Px)'(t) + (I - P'(t))(Qx)(t)$.

Now we see that the growth of solutions is mainly determined by the properties of the nonlinear inherent regular ODE (3.36) . Based on this we will derive estimates for the difference of two solutions of (3.33) or, more general, the difference of a solution of (3.33) and the solution of a perturbed DAE.
In the following let $x(\cdot)$ denote a solution of (3.33) with initial value $x_0 = x(t_0)$ and $\bar{x}(\cdot)$ denote a solution of (3.33) which is perturbed by a right-hand-side $q(\cdot)$ with initial value $\bar{x}_0 = \bar{x}(t_0)$, i.e.

$$\begin{aligned} f((Px)'(t) - P'(t)x(t), x(t), t) &= 0 \quad , \quad x(t_0) = x_0 \quad , \\ f((P\bar{x})'(t) - P'(t)\bar{x}(t), \bar{x}(t), t) &= q(t) \quad , \quad \bar{x}(t_0) = \bar{x}_0 \quad . \end{aligned} \tag{3.38}$$

Further, let denote

$$\begin{aligned} u_0 &= (Px)(t_0) \quad , \quad w_0 = (Px)' - P'x)(t_0) + (Qx)(t_0) \\ \bar{u}_0 &= (P\bar{x})(t_0) \quad , \quad \bar{w}_0 = (P\bar{x})' - P'\bar{x})(t_0) + (Q\bar{x})(t_0) \end{aligned}$$

and let

$$\begin{aligned} w = w(u,t) \quad &\text{solve} \quad f(w, u + Q(t)w, t) = 0 \quad , \quad w(u_0, t_0) = w_0 \quad \text{and} \\ w^q = w^q(u,t) \quad &\text{solve} \quad f(w, u + Q(t)w, t) = q(t) \ , \ w^q(\bar{u}_0, t_0) = \bar{w}_0 \quad . \end{aligned}$$

Then , we know from theorem 3.12 that

$$x(t) = u(t) + Q(t)w(u(t), t) \tag{3.39}$$

$$\bar{x}(t) = \bar{u}(t) + Q(t)w^q(\bar{u}(t), t) \quad , \tag{3.40}$$

where $u(\cdot)$ and $\bar{u}(\cdot)$ solve the inherent regular ODEs

$$u' = P'u + P(I + P')w(u,t) := g(u,t) \quad , \quad u(t_0) = u_0 \quad , \tag{3.41}$$

$$\bar{u}' = P'\bar{u} + P(I + P')w^q(\bar{u},t) := \bar{g}(\bar{u},t) \quad , \quad \bar{u}(t_0) = \bar{u}_0 \quad . \tag{3.42}$$

We start estimating the difference $w(u,t) - w^q(u,t)$ for fixed values (u,t).

Lemma 3.13 *Let the DAE (3.32) have (tractability-) index 1 and let the inverse of $G(y,x,t)$ and also the inverse of mean-values $\int_0^1 G(sy_1 + (1-s)y_2, sx_1 + (1-s)x_2, t)ds$ be uniformly bounded by some constant $c_{G^{-1}}$. Let w and w^q be defined as above. Then it holds*

$$|w(u,t) - w^q(u,t)| \le c_{G^{-1}} \cdot |q(t)| \tag{3.43}$$

for all (u,t) where w and w^q are defined.

Proof: Let (u,t)be an arbitrary but fixed value such that w and w^q are defined in (u,t) and let denote for short $w = w(u,t)$, and $w^q = w^q(u,t)$. We define the function

$$F(w) := f(w, u + Q(t)w, t) \quad .$$

Then it holds $\quad F(w) = 0 \quad$ and $\quad F(w^q) = q(t) \quad$. Hence, it follows

$$\begin{aligned} q(t) = F(w^q) - F(w) &= \int_0^1 F'_w(sw^q + (1-s)w)ds(w^q - w) \\ &= \int_0^1 G(sw^q + (1-s)w, u + Q(t)(sw^q + (1-s)w), t)ds(w^q - w) \end{aligned}$$

and

$$w^q - w = (\int_0^1 G(sw^q + (1-s)w, u + Q(t)(sw^q + (1-s)w), t)ds)^{-1} q(t)$$

which implies the assertion. q.e.d.

The next step is to compare the solutions $u(\cdot)$ and $\bar{u}(\cdot)$ of the inherent regular ODEs (3.41) and (3.42).

Lemma 3.14 *Let $u(\cdot)$ and $\bar{u}(\cdot)$ be the solutions of the inherent regular ODEs (3.41) and (3.42) and let hold the suppositions of lemma 3.13. Then it holds for all $t \geq t_0$*

$$|u(t) - \bar{u}(t)| \leq e^{L(t)} \cdot |u(t_0) - \bar{u}(t_0)| + e^{L(t)} \int_{t_0}^{t} e^{-L(s)} c_q(s) ds \quad , \tag{3.44}$$

where

$$L(t) = \int_{t_0}^{t} \max_{v \in [u(\tau), \bar{u}(\tau)]} \mu^{\operatorname{im} P(\tau)}[-M(w(v,\tau), v + Q(\tau)w(v,\tau), \tau)] d\tau \quad ,$$

$$c_q(s) = \|(P + P'Q)(s)\| \cdot c_{G^{-1}} \cdot |q(s)| \quad , \quad \textit{and}$$

$$M(y,x,t) = -P'(t)P_{can}(y,x,t) + P(t)G^{-1}(y,x,t)B(y,x,t)P(t) \quad .$$

Proof: We consider the scalar function $m(t) := |u(t) - \bar{u}(t)|$. The assertion follows by the fundamental lemma if it is shown that $m(t)$ satisfies the following differential inequality:

$$D_+ m(t) \quad \leq \quad \max_{v \in [u(t), \bar{u}(t)]} \mu^{\operatorname{im} P(t)}[-M(w(v,t), v + Q(t)w(v,t), t)] \cdot m(t) + c_q(t) \quad ,$$

where $D_+m(t)$ denotes the Dini-derivative $D_+m(t) := \liminf_{h\to 0^+} \frac{m(t+h)-m(t)}{h}$. But, it holds:

$$
\begin{aligned}
m(t+h) &= |u(t+h)-\bar{u}(t+h)| \\
&= |u(t)-\bar{u}(t)+h(u'(t)-\bar{u}'(t))|+o(h) \\
&= |u(t)-\bar{u}(t)+h(g(u(t),t)-\bar{g}(\bar{u}(t),t)|+o(h) \\
&\leq |u(t)-\bar{u}(t)+h(g(u(t),t)-g(\bar{u}(t),t)|+h|g(\bar{u}(t),t)-\bar{g}(\bar{u}(t),t)|+o(h) \\
&= |u(t)-\bar{u}(t)+h\int_0^1 g'_u(su(t)+(1-s)\bar{u}(t),t)ds(u(t)-\bar{u}(t))| \\
&\qquad\qquad +h|(P(I+P'))(t)(w(\bar{u}(t),t)-w^q(\bar{u}(t),t))|+o(h) \\
&\leq \|I+h\int_0^1 g'_u(su(t)+(1-s)\bar{u}(t),t)ds\|^{\operatorname{im}P}|u(t)-\bar{u}(t)| \\
&\qquad\qquad +h\|(P+P'Q)(t)\|\cdot|w(\bar{u}(t),t)-w^q(\bar{u}(t),t)|+o(h) \\
&\leq \|I+h\int_0^1 g'_u(su(t)+(1-s)\bar{u}(t),t)ds\|^{\operatorname{im}P}m(t) \\
&\qquad\qquad +h\underbrace{\|(P+P'Q)(t)\|\cdot c_{G^{-1}}\cdot|q(t)|}_{c_q(t)}+o(h)
\end{aligned}
$$

This gives in the limit

$$
D_+m(t) \;:=\; \liminf_{h\to 0^+}\frac{m(t+h)-m(t)}{h} \;\leq\; \max_{v\in[u(t),\bar{u}(t)]}\mu^{\operatorname{im}P(t)}[g'_u(v,t)]\cdot m(t)+c_q(t) \quad .
$$

It remains to check that

$$
\mu^{\operatorname{im}P}[g'_u(v,t)] = \mu^{\operatorname{im}P}[(P'P_{can}-PG^{-1}BP)(w(v,t),v+Q(t)w(v,t),t)] \quad .
$$

But, by the definition of g as the right-hand-side of the inherent regular ODE (3.41) it is

$$
g'_u(u,t) = P'(t)+(P(I+P'))(t)w'_u(u,t)
$$

and $w'_u(u,t)$ is determined by the implicit nonlinear equation (3.34): It holds with dropped arguments $w'_u = -(A+BQ)^{-1}B$ and

$$
\begin{aligned}
g'_u\cdot P &= (P'-(P+P'P)G^{-1}B)\cdot P \\
&= (P'-(P+P'Q)G^{-1}B)\cdot P \\
&= P'(P-\underbrace{QG^{-1}B}_{Q_{can}}\cdot P)-PG^{-1}BP \\
&= P'P_{can}-PG^{-1}BP = -M \quad .
\end{aligned}
$$

The last equality follows from (3.16) (see remark 3.1). This completes the proof.

q.e.d.

Taking into account the composition of the solutions as a sum of differential and algebraic components we are now able to give estimates of the solutions.

Theorem 3.15 *Let x and $\bar{x}$ be solutions of the original nonlinear DAE (3.32) and the perturbed one (3.38).Let the DAE (3.32) have (tractability-) index 1 and let hold the supposition of lemma 3.13 and the notations of lemma 3.14. Then it holds for all $t \geq t_0$*

$$\begin{aligned} |x(t)-\bar{x}(t)| \quad \leq \quad & \max_{(w,v)} \|P_{can}(w,v,t)\|^{\operatorname{im} P(t)} \cdot (e^{L(t)} \cdot |(Px - P\bar{x})(t_0)| + e^{L(t)} \int_{t_0}^{t} e^{-L(s)} c_q(s) ds) \\ & + \|Q(t)\| c_{G^{-1}} |q(t)| \quad . \end{aligned} \tag{3.45}$$

Proof: From (3.39),(3.40) we obtain with dropped argument t

$$\begin{aligned} |x-\bar{x}| &= |u-\bar{u}+Q(w(u,\cdot)-w^q(\bar{u},\cdot))| \\ &= |u-\bar{u}+Q(w(u,\cdot)-w(\bar{u},\cdot))+Q(w(\bar{u},\cdot)-w^q(\bar{u},\cdot))| \\ &\leq |u-\bar{u}+Q(w(u,\cdot)-w(\bar{u},\cdot))| + \|Q\| \cdot c_{G^{-1}}|q| \\ &= |(I+Q\int_0^1 w'_u(su+(1-s)\bar{u})ds)(u-\bar{u})| + \|Q\| \cdot c_{G^{-1}}|q| \\ &= |(I+Q\int_0^1 -(G^{-1}B)(y_s,x_s,\cdot)ds)(u-\bar{u})| + \|Q\| \cdot c_{G^{-1}}|q| \\ &= |(I-\int_0^1 (\underbrace{QG^{-1}B}_{Q_{can}})(y_s,x_s,\cdot)ds)(u-\bar{u})| + \|Q\| \cdot c_{G^{-1}}|q| \\ &= |\int_0^1 P_{can}(y_s,x_s,\cdot)ds(u-\bar{u})| + \|Q\| \cdot c_{G^{-1}}|q| \\ &\leq \max_{(w,v)} \|P_{can}(w,v,\cdot)\|^{\operatorname{im} P}|u-\bar{u}| + \|Q\| \cdot c_{G^{-1}}|q| \quad . \end{aligned}$$

Here we used the abbreviations $y_s = w(su+(1-s)\bar{u})$ and $x_s = su+(1-s)\bar{u}+Qw(su+(1-s)\bar{u})$. The assertion now follows with the estimate (3.44) from lemma 3.14 . q.e.d.

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